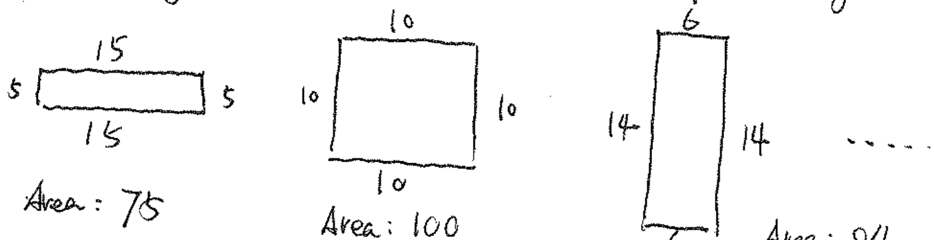
§3.7. Optimization

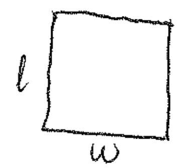
- Goal: Find abs max/min in real life
- Motivation: Fence problem: Suppose we have 40 meters of fence to make a rectangular corral. What length and width will fence off the largest area?

Possible figures:



Area: 75 Area: 100 Area: 84

Math model: Give a rectangle with length l and width w.
Suppose the perimeter is constant, i.e., $2l+2W=40$.
What are the values of l and w that make the area $A=l\cdot w$ reach its maximum?



Perimeter: $2l+2W$
Area: $l\cdot W$

Optimazation problem: Give the constrain $2l+2W=40$. Find the l and w to maximize the function $A=l\cdot w$.

Key step: Use constraint to convert A into a one variable function.

$$2l+2W=40 \Rightarrow l+W=20 \Rightarrow W=20-l \Rightarrow A=l(20-l)$$

Variable: l. Function: $A=l(20-l)=20l-l^2$. Domain: $[0,20]$

Now use the method in $\boxed{\S 3.1}$ to find the abs max of A over $[0,20]$.

Critical points: $A'=(20l-l^2)'=20-2l=0 \Rightarrow l=10$

endpoints: $l=0$, $l=20$

List the values of A at the above points: $A(10)=20\cdot 10-10^2=100$
$A(0)=0$, $A(20)=0$

i.e. The abs max of A is 100, it is attained at $l=10$.

i.e. The area is maximized at with dimension $l=10$, $W=10$
The maximum area is 100.

Method for optimization:

Step 1: Draw the picture with all varying quantities. Find the target quantity (target function) to be maximized or minimized.

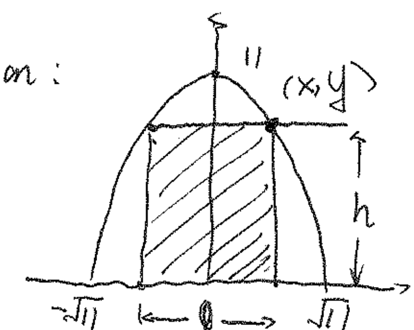
Step 2: Find the constraint (the relations/equations relating variables). Choose one appropriate quantity (variable) and use the constraint to express all the ~~oth~~ other quantities in terms of this main quantity (controlling variable). Express the target function only using this main variable and find its domain.

Step 3: Find the abs max/min via method in §3.1 (critical + endpoints). Draw the conclusion. when the function is maximized/minimized.

eg.1. (s'6). A rectangle is inscribed with its base on x-axis and its upper corners on the parabola $y=11-x^2$. What are the dimensions of such a rectangle with the greatest possible area?

WARNING: The problem is not asking for the max of $y=11-x^2$.

Solution:



Draw the curve $y=11-x^2$ first. y intercept: $y=11$

x intercepts: $x=\pm\sqrt{11}$

$-x^2$ (negative coefficient) $\Rightarrow$ opens down.

even function, symmetric about y-axis.

Dimension of the inscribed rectangle: width w, height h, Area $A=w\cdot h$

Constraint: Upper corner on the parabola: w, h should be related to $y=11-x^2$.

Right upper corner with coordinate: $(x,y) \Rightarrow h=y$, $w=2x$.

$h=11-x^2$,

$A(x)=2x\cdot(11-x^2)$. Domain: $0\le x\le\sqrt{11}$.

$=22x-2x^3$.

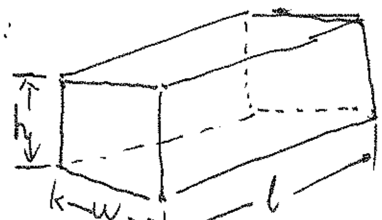
(§3.1). $A'(x)=22-6x^2=0 \Rightarrow x^2=\frac{22}{6}=\frac{11}{3} \Rightarrow x=\sqrt{\frac{11}{3}}$ (negative solution is discarded)

$A(\sqrt{\frac{11}{3}})=2\cdot\sqrt{\frac{11}{3}}\,(11-\frac{11}{3})=\frac{44}{3}\cdot\sqrt{\frac{11}{3}}$ (abs max) at $x=\sqrt{\frac{11}{3}}$.

Dimension: $w=2x=2\sqrt{\frac{11}{3}}$, $h=11-x^2=11-\frac{11}{3}=\frac{22}{3}$

e.g.2. (Fall 15). A total of $1200\ cm^2$ of material is to be used to make a box with no top. Assume that the base is to be twice as long as its wide. Find the largest volume of such a box.

Picture:



Dimension: height h, weight w, length l

Volume (Target function): $V = h\cdot w\cdot l \cdots (1)$

Surface area (without top) is a constraint:

$2h\cdot w + 2h\cdot l + w\cdot l = 1200 \quad \cdots\cdots (2)$

Relation between length and width: $l = 2w \cdots (3)$

Goal: choose w as the main variable and express l, h, V by w.

$l = 2w \Rightarrow 2\cdot h\cdot w + 2h\cdot 2w + w\cdot 2w = 1200$ (Plug $l=2w$ into (2))

$\Rightarrow 6h\cdot w + 2w^2 = 1200$ (Fix w, solve for h)

$\Rightarrow h = \frac{1200-2w^2}{6w}$

Plug $l = 2w$, $h = \frac{1200-2w^2}{6w}$ into $V = h\cdot w\cdot l$

$V(w) = \frac{1200-2w^2}{6w}\cdot w\cdot 2w \overset{\text{simplify}}{=\!=} \left(\frac{1200-2w^2}{3}\right)\cdot w = \frac{1200}{3}w - \frac{2}{3}w^3 = \boxed{400w - \frac{2}{3}w^3}$

Now use the method in §3.1 to find $V(w)$'s abs max and the corresponding w.

☆ Notice all w, l, h have to be positive, $h = \frac{1200-2w^2}{6w} > 0 \Rightarrow 1200 - 2w^2 < 0 \Rightarrow w^2 < 600$

Domain of $V(w)$: $w \in [0, 10\sqrt{6}]$, $\Rightarrow w < \sqrt{600} = \sqrt{6}\cdot 10$

Critical points of V: $V' = (400w - \frac{2}{3}w^3)' = 400 - \frac{2}{3}\cdot 3w^2 = 0$

$\Rightarrow 400 - 2\cdot w^2 = 0 \Rightarrow w^2 = 200 \Rightarrow w = \sqrt{200} = 10\cdot\sqrt{2} \in [0, 10\sqrt{6})$

Endpoints: $w=0$, $w = \cancel{10\sqrt{6}}\ \sqrt{600} \Rightarrow V(0) = 0$, $V(\sqrt{600}) = 400\cdot\sqrt{600} - \frac{2}{3}(\sqrt{600})^3 = 0$.

$V(\sqrt{200}) = 400\cdot\sqrt{200} - \frac{2}{3}(\sqrt{200})^3 = 400\cdot\sqrt{200} - \frac{2}{3}(\sqrt{200})^2\cdot\sqrt{200}$

$= 400\sqrt{200} - \frac{2}{3}\cdot 200\cdot\sqrt{200} = \boxed{\frac{800}{3}\cdot\sqrt{200}} = \frac{8000}{3}\sqrt{2}$

The abs max $= \frac{800}{3}\sqrt{200}$,

attained at $w = \sqrt{200}$ $\left(\Rightarrow l = 2\cdot\sqrt{200},\ h = \frac{1200 - 2\cdot(\sqrt{200})^2}{6\cdot\sqrt{200}} = \frac{800}{6\sqrt{200}}\right)$

§ Newton's Method

- Formula: Newton's Method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. ....(1)
- The above formula is a numerical method for finding successively better approximations to the roots (or zeros) of a function $f(x)$, i.e., the solutions x to the equation $f(x)=0$.

☆ Give $f(x)$, $f'(x)$ and INITIAL value x_1, the formula (1) runs inductively for $n=1,2,3,\ldots$

$n=1$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$; $n=2$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$; ...... (2)

Conclusion: The list of numbers given by (2), $x_1, x_2, x_3, \ldots$ are getting closer and closer (approaching) the solutions of $f(x)=0$.

☆ Key point: know how to use (2) to compute x_2, x_3.

e.g.1 Suppose you are using Newton's method to estimate the value of $\sqrt{2}$
(s16) by finding the root of $x^2-2=0$. If you use $x_1=1$, find x_2.

Solution:
Preparation: $f(x)=x^2-2$, $f'(x)=2x$, $x_1=1$. Plug into (2).

$$(2) \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^2-2}{2\cdot 1} = 1 - \frac{-1}{2} = \boxed{\frac{3}{2}}$$

Remark: If the problem asks for x_3, then plug $x_2=\frac{3}{2}$ into (2) one more time

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{2} - \frac{f(\frac{3}{2})}{f'(\frac{3}{2})} = \frac{3}{2} - \frac{(\frac{3}{2})^2-2}{2\cdot\frac{3}{2}} = \frac{3}{2} - \frac{\frac{1}{4}}{3} = \frac{17}{12}$$

e.g.2. Use Newton's method to approximate a nonzero solution of the equation $4\sin x = x$. Let $x_1=2$. Find x_2 to four decimal places (use calculator)

Key step: Rewrite $4\sin x = x$ as $f(x)=0$. $4\sin x - x = 0$, i.e., $f(x)=4\sin x - x$.

Solution:
$f'(x)=4\cos x - 1$, $x_1=2 \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{4\sin 2 - 2}{4\cos 2 - 1} \approx 2.6144$.